5

One-dimensional flow: mechanism for conservation of flow properties

General flows are three dimensional, but many of them may be studied as if they are one dimensional. For example, whenever a flow in a tube is considered, if it is studied in terms of mean velocity, it is a one-dimensional flow, which is studied very simply. Presented below are the methods of solution of those cases which may be studied as one-dimensional flows by using the continuity equation, energy equation and momentum equation.

5.1 Continuity equation

In steady flow, the mass flow per unit time passing through each section does not change, even if the pipe diameter changes. This is the law of conservation of mass.

For the pipe shown in Fig. 5.1 whose diameter decreases between sections 1 and 2, which have cross-sectional areas A_1 and A_2 respectively, and at which the mean velocities are v_1 and v_2 and the densities ρ_1 and ρ_2 respectively,

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

namely,

$$\rho A v = \text{constant} \tag{5.1}$$

If the fluid is incompressible, e.g. water, with ρ being effectively constant, then

$$Av = \text{constant} \tag{5.2}$$

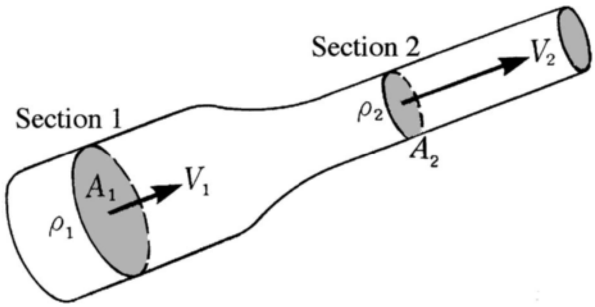


Fig. 5.1 Mass flow rate passing through any section is constant

ρAv is the mass of fluid passing through a section per unit time and this is called the mass flow rate. Av is that volume and this is called the volumetric flow rate, which is therefore constant is an incompressible pipe flow.

Equations (5.1) and (5.2) state that the flow is continuous, with no loss or gain, so these equations are called the continuity equations. They are an expression of the principle of conservation of mass when applied to fluid flow. It is clear from eqn (5.1) that the flow velocity is inversely proportional to the cross-sectional area of the pipe. When the diameter of the pipe is reduced, the flow velocity increases.

5.2 Conservation of energy

5.2.1 Bernoulli's equation

Consider a roller-coaster running with great excitement in an amusement park (Fig. 5.2). The speed of the roller-coaster decreases when it is at the top of the steep slope, and it increases towards the bottom. This is because the potential energy increases and kinetic energy decreases at the top, and the opposite occurs at the bottom. However, ignoring frictional losses, the sum of the two forms of energy is constant at any height. This is a manifestation of the principle of conservation of energy for a solid.

Figures 5.3(a) and (b) show the relationship between the potential energy of water (its level) and its kinetic energy (the speed at which it gushes out of the pipe).

A fluid can attain large kinetic energy when it is under pressure as shown in Fig. 5.3(c). A water hydraulic or oil hydraulic press machine is powered by the forces and energy due to such pressure.

In fluids, these three forms of energy are exchangeable and, again ignoring frictional losses, the total energy is constant. This is an expression of the law of conservation of energy applied to a fluid.

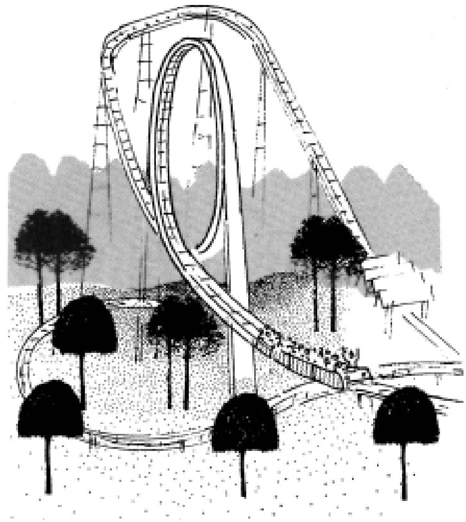


Fig. 5.2 Movement of roller-coaster

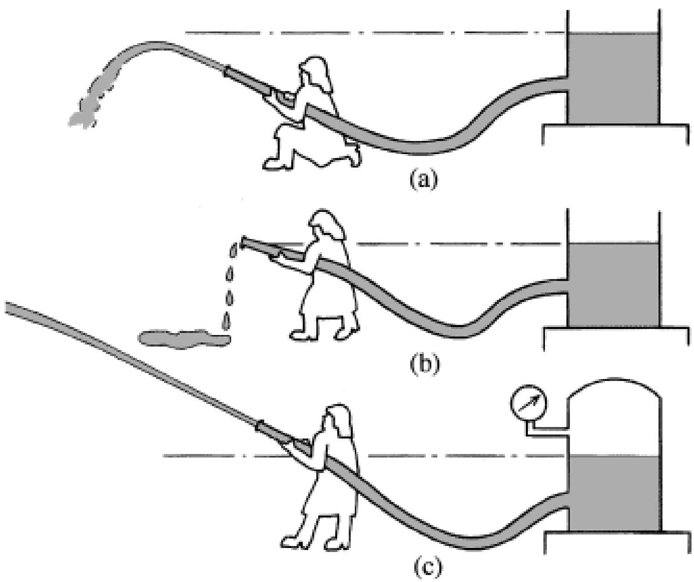


Fig. 5.3 Conservation of fluid energy

A streamline (a line which follows the direction of the fluid velocity) is chosen with the coordinates shown in Fig. 5.4. Around this line, a cylindrical element of fluid having the cross-sectional area $\mathrm{d}A$ and length $\mathrm{d}s$ is considered. Let p be the pressure acting on the lower face, and pressure $p + (\partial p/\partial s)\mathrm{d}s$ acts on the upper face a distance $\mathrm{d}s$ away. The gravitational force acting on this element is its weight, $\rho g\,\mathrm{d}A\,\mathrm{d}s$. Applying Newton's second

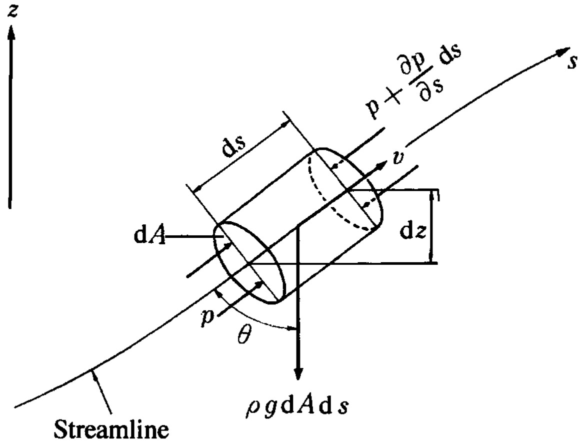


Fig. 5.4 Force acting on fluid on streamline

law of motion to this element, the resultant force acting on it, and producing acceleration along the streamline, is the force due to the pressure difference across the streamline and the component of any other external force (in this case only the gravitational force) along the streamline.

Therefore the following equation is obtained:

$$\rho\,\mathrm{d}A\,\mathrm{d}s\frac{\mathrm{d}v}{\mathrm{d}t} = -\mathrm{d}A\frac{\partial p}{\partial s}\mathrm{d}s - \rho g\,\mathrm{d}A\,\mathrm{d}s\cos\theta$$

or

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial s} - g\cos\theta \tag{5.3}$$

The velocity may change with both position and time. In one-dimensional flow it therefore becomes a function of distance and time, $v = v(s, t)$. The change in velocity $\mathrm{d}v$ over time $\mathrm{d}t$ may be written as

$$\mathrm{d}v = \frac{\partial v}{\partial t}\mathrm{d}t + \frac{\partial v}{\partial s}\mathrm{d}s$$

The acceleration is then

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial t}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}$$

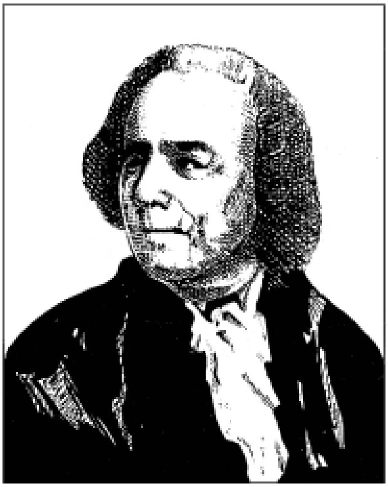
If the z axis is the vertical direction as shown in Fig. 5.4, then

$$\cos\theta = \mathrm{d}z/\mathrm{d}s$$

So eqn (5.3) becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\mathrm{d}z}{\mathrm{d}s} \tag{5.4}$$

In the steady state, $\partial v/\partial t = 0$ and eqn (5.4) would then become



Leonhard Euler (1707–83)
Mathematician born near Basle in Switzerland. A pupil of Johann Bernoulli and a close friend of Daniel Bernoulli. Contributed enormously to the mathematical development of Newtonian mechanics, while formulating the equations of motion of a perfect fluid and solid. Lost his sight in one eye and then both eyes, as a result of a disease, but still continued his research.

$$v\frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}s} - g\frac{\mathrm{d}z}{\mathrm{d}s} \tag{5.5}$$

Equation (5.4) or (5.5) is called Euler's equation of motion for one-dimensional non-viscous fluid flow. In incompressible fluid flow with two unknowns (v and p), the continuity equation (5.2) must be solved simultaneously. In compressible flow, ρ becomes unknown, too. So by adding a third equation of state for a perfect gas (2.14), a solution can be obtained.

Equation (5.5) is integrated with respect to s to obtain a relationship between points a finite distance apart along the streamline. This gives

$$\frac{v^2}{2} + \int\frac{\mathrm{d}p}{\rho} + gz = \text{constant} \tag{5.6}$$

and for an incompressible fluid (ρ = constant),

$$\frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant} \tag{5.7}$$

between arbitrary points, and therefore at all points, along a streamline.

Dividing each term in eqn (5.7) by g,

$$\frac{v^2}{2g} + \frac{p}{\rho g} + z = H = \text{constant} \tag{5.8}$$

Multiplying each term of eqn (5.7) by ρ,

$$\frac{\rho v^2}{2} + p + \rho gz = \text{constant} \tag{5.9}$$

The units of the terms in eqn (5.7) are $\mathrm{m^2/s^2}$, which can be expressed as $\mathrm{kg\,m^2/(s^2\,kg)}$. Since $\mathrm{kg\,m^2/s^2} = \mathrm{J}$ (for energy), then $v^2/2, p/\rho$ and gz in eqn

Daniel Bernoulli (1700–82)
Mathematician born in Groningen in the Netherlands. A good friend of Euler. Made efforts to popularise the law of fluid motion, while tackling various novel problems in fluid statics and dynamics. Originated the Latin word *hydrodynamica*, meaning fluid dynamics.

(5.7) represent the kinetic energy, energy due to pressure and potential energy respectively, per unit mass.

The terms of eqn (5.8) represent energy per unit weight, and they have the units of length (m) so they are commonly termed heads.

$$\frac{v^2}{2g}: \quad \text{velocity head}$$
$$\frac{p}{\rho g}: \quad \text{pressure head}$$
$$z: \quad \text{potential head}$$
$$H: \quad \text{total head}$$

The units of the terms of eqn (5.9) are $\text{kg}/(\text{s}^2\,\text{m})$ expressing energy per unit volume. Thus, eqns (5.7) to (5.9) express the law of conservation of energy in that the sum of the kinetic energy, energy due to pressure and potential energy (i.e. the total energy) is always constant. This is Bernoulli's equation.

If the streamline is horizontal, then the term ρgh can be omitted giving the following:

$$\frac{\rho v^2}{2} + p_s = p_t \tag{5.10}$$

where $\rho v^2/2$ is called the dynamic pressure, p_s the static pressure, and p_t the total pressure or stagnation pressure.

Static pressure p_s can be detected, as shown in Fig. 5.5, by punching a small hole vertically in the solid wall face parallel to the flow.

As Bernoulli's theorem applies to a flow line, it is also applicable to the flow in a pipe line as shown in Fig. 5.6. Assume the pipe line is horizontal, and $z_1 = z_2$ in eqn (5.8). The following relative equation is obtained:

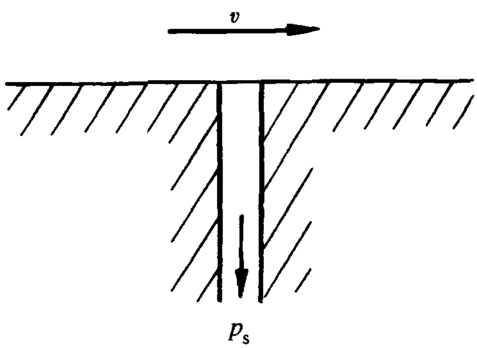


Fig. 5.5 Picking out of static pressure

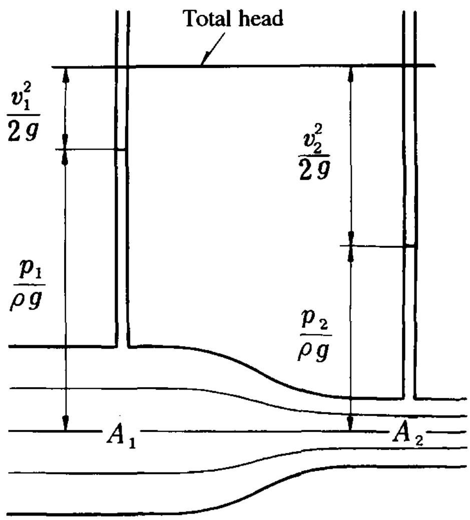


Fig. 5.6 Exchange between pressure head and velocity head

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} = \frac{v_2}{2g} + \frac{p_2}{\rho g} \tag{5.11}$$

Also, from the continuity equation,

$$v_1 A_1 = v_2 A_2 \tag{5.12}$$

Consequently, whenever $A_1 > A_2$, then $v_1 < v_2$ and $p_1 > p_2$. In other words, where the flow channel is narrow (where the streamlines are dense), the flow velocity is large and the pressure head is low.

As shown in Fig. 5.7, whenever water flows from tank 1 to tank 2, the energy equations for sections 1, 2 and 3 are as follows from eqn (5.8):

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} + z_1 = \frac{v_2^2}{2} + \frac{p_2}{\rho} + z_2 + h_2 = \frac{v_3^2}{2} + \frac{p_3}{\rho} + z_3 + h_3 \tag{5.13}$$

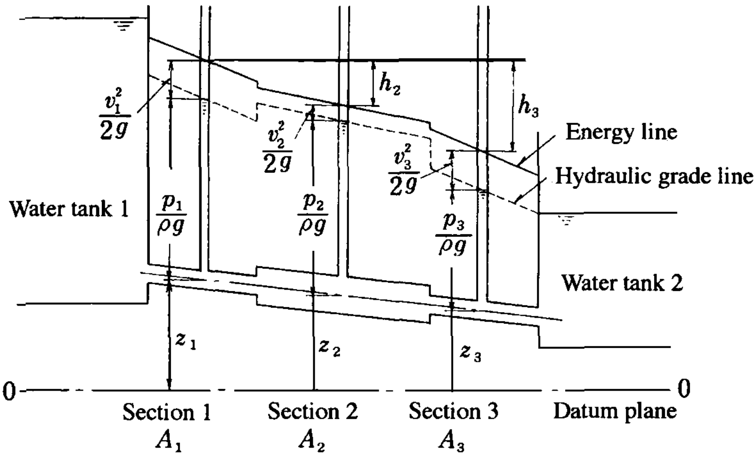


Fig. 5.7 Hydraulic grade line and energy line

h_2 and h_3 are the losses of head between section 1 and either of the respective sections.

In Fig. 5.7, the line connecting the height of the pressure heads at respective points of the pipe line is called the hydraulic grade line, while that connecting the heights of all the heads is called the energy line.

5.2.2 Application of Bernoulli's equation

Various problems on the one-dimensional flow of an ideal fluid can be solved by jointly using Bernoulli's theorem and the continuity equation.

Venturi tube

As shown in Fig. 5.8, a device where the flow rate in a pipe line is measured by narrowing a part of the tube is called a Venturi tube. In the narrowed part of the tube, the flow velocity increases. By measuring the resultant decreasing pressure, the flow rate in the pipe line can be measured.

Let A be the section area of the Venturi tube, v the velocity and p the pressure, and express the states of sections 1 and 2 by subscripts 1 and 2 respectively. Then from Bernoulli's equation

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

Assuming that the pipe line is horizontal,

$$z_1 = z_2$$

$$\frac{v_2^2 - v_1^2}{2g} = \frac{p_1 - p_2}{\rho g}$$

From the continuity equation,

$$v_1 = v_2 A_2 / A_1$$

Giovanni Battista Venturi (1746–1822)
Italian physicist. After experiencing life as a priest, teacher and auditor, finally became a professor of experimental physics. Studied the effects of eddies and the flow rates at various forms of mouthpieces fitted to an orifice, and clarified the basic principles of the Venturi tube and the hydraulic jump in an open water channel.

Therefore,

$$v_2 = \frac{1}{\sqrt{1-(A_2/A_1)^2}}\sqrt{2g\frac{p_1-p_2}{\rho g}} \tag{5.14}$$

and

$$\frac{p_1-p_2}{\rho g} = H$$

Consequently, the flow rate

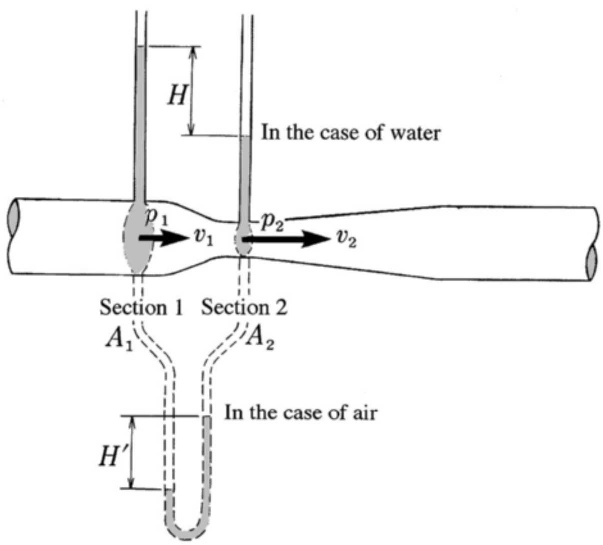


Fig. 5.8 Venturi tube

Henry de Pitot (1695–1771)
Born in Aramon in France. Studied mathematics and physics in Paris. As a civil engineer, undertook the drainage of marshy lands, construction of bridges and city water systems, and flood countermeasures. His books cover structures, land survey, astronomy, mathematics, sanitary equipment and theoretical ship steering in addition to hydraulics. The famous Pitot tube was announced in 1732 as a device to measure flow velocity.

$$Q = A_2 v_2 = \frac{A_2}{\sqrt{1-(A_2/A_1)^2}}\sqrt{2gH} \tag{5.15}$$

In the case where the flowing fluid is a gas, $p_1 - p_2$ is measured by a U-tube.

However, since there is some loss of energy between sections A_1 and A_2 in actual cases, the above equation is amended as follows:

$$Q = C\frac{A_2}{\sqrt{1-(A_2/A_1)^2}}\sqrt{2gH} \tag{5.16}$$

C is called the coefficient of discharge. It is determined through experiment. Equation (5.16) is also applicable to the case where the tube is inclined.

Pitot tube

Pitot, who was engaged in research work, hit upon an idea one day for a very simple measuring device of flow rate. It was a device where the lower end of a glass tube is bent by 90° and supported against the flow. The flow velocity was to be measured by measuring the increased height of the water level. It is said that, as soon as he had hit upon this idea, he rushed to the River Seine carrying a glass tube with a bent end. The result of an experiment as shown in Fig. 5.9 confirmed his expectation. The device incorporating that idea is shown in Fig. 5.10. This device is called a Pitot tube, and it is widely used even nowadays.

The tube is so designed that at the streamlined end a hole is opened in the face of the flow, while another hole in the direction vertical to the flow is used in order to pick out separate pressures.

Let p_A and v_A respectively be the static pressure and the velocity at position A of the undisturbed upstream flow. At opening B of the Pitot tube, the flow is stopped, making the velocity zero and the pressure p_B. B is called the stagnation point. Apply Bernoulli's equation between A and B,

Fig. 5.9 Pitot's first experiment

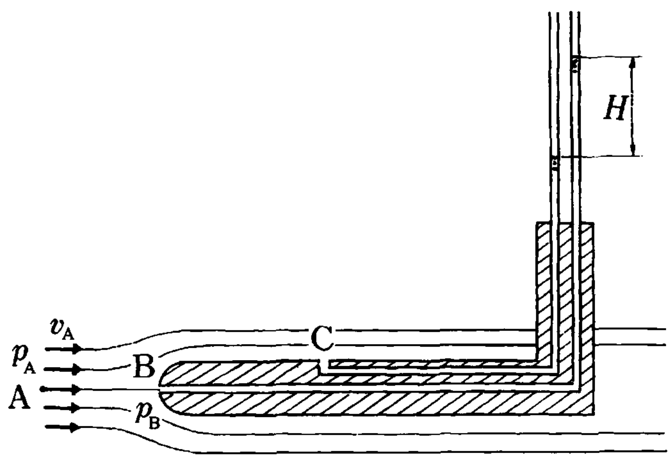


Fig. 5.10 Pitot tube

and

$$\frac{p_{\mathrm{A}}}{\rho g}+\frac{v_{\mathrm{A}}^2}{2g}=\frac{p_{\mathrm{B}}}{\rho g}$$

or

$$v_{\mathrm{A}}=\sqrt{2g\frac{p_{\mathrm{B}}-p_{\mathrm{C}}}{\rho}} \tag{5.17}$$

In a parallel flow, the static pressure p_{A} is the same on the streamline adjacent to A and is detected by hole C normal to the flow. Thus, since $p_{\mathrm{C}}=p_{\mathrm{A}}$, eqn (5.17) becomes:

$$v_{\mathrm{A}}=\sqrt{2\frac{p_{\mathrm{B}}-p_{\mathrm{C}}}{\rho}} \tag{5.18}$$

And, since $(p_{\mathrm{B}}-p_{\mathrm{C}})/\rho g=H$, the following equation is obtained:

$$v_{\mathrm{A}}=\sqrt{2gH} \tag{5.19}$$

In the case where the flowing fluid is a gas, $p_{\mathrm{B}}-p_{\mathrm{C}}$ is measured with a U-tube.

However, with an actual Pitot tube, since some loss occurs due to its shape and the fluid viscosity, the equation is modified as follows:

$$v_{\mathrm{A}}=C_{\mathrm{v}}\sqrt{2gH} \tag{5.20}$$

where C_{v} is called the coefficient of velocity.

Flow through a small hole 1: *the case where water level does not change*

As shown in Fig. 5.11, we study here the case where water is discharging from a small hole on the side of a water tank. Such a hole is called an orifice. As

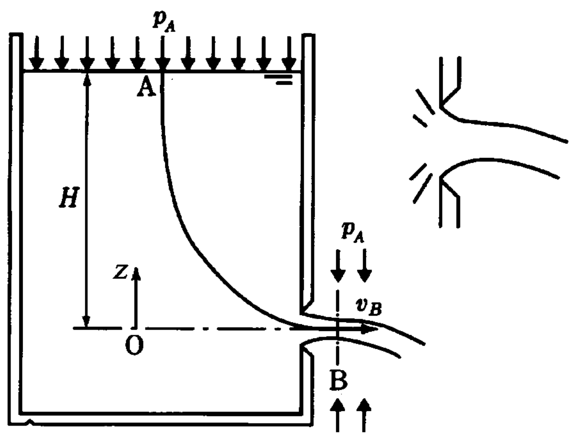


Fig. 5.11 Flow through a small hole (1)

shown in the figure, the spouting flow contracts to have its smallest section B a small distance from the hole. Here, it is conceived that the flow lines are almost parallel so that the pressures are uniform from the periphery to the centre of the flow. This part of the flow is called the vena contracta.

Assume that fluid particle A on the water surface has flowed down to section B. Then, from Bernoulli's theorem,

$$\frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A = \frac{p_A}{\rho g} + \frac{v_B^2}{2g} + z_B$$

Assuming that the water tank is large and the water level does not change, at point A, $v_A = 0$ and $z_A = H$, while at point B, $z_B = 0$. If p_A is the atmospheric pressure, then

$$\frac{p_A}{\rho g} + H = \frac{p_A}{\rho g} + \frac{v_B^2}{2g}$$

or

$$v_B = \sqrt{2gH} \tag{5.21}$$

Equation (5.21) is called Torricelli's theorem.

Coefficient of contraction Ratio C_c of area a_c of the smallest section of the discharging flow to area a of the small hole is called the coefficient of contraction, which is approximately 0.65:

$$a_c = C_c a \tag{5.22}$$

Coefficient of velocity The velocity of spouting flow at the smallest section is less than the theoretical value $\sqrt{2gH}$ produced by the fluid velocity and the edge of the small hole. Ratio C_v of actual velocity v to $\sqrt{2gH}$ is called the coefficient of velocity, which is approximately 0.95:

$$v = C_v v_B = C_v\sqrt{2gH} \tag{5.23}$$

Coefficient of discharge Consequently, the actual discharge rate Q is

$$Q = C_c a\, C_v v_B = C_c C_v a\sqrt{2gH} \tag{5.24}$$

Furthermore, setting $C_c C_v = C$, this can be expressed as follows:

$$Q = Ca\sqrt{2gH} \tag{5.25}$$

C is called the coefficient of discharge. For a small hole with a sharp edge, C is approximately 0.60.

Flow through a small hole 2: *the case where water level changes*
The theoretical flow velocity is

$$v = \sqrt{2gH}$$

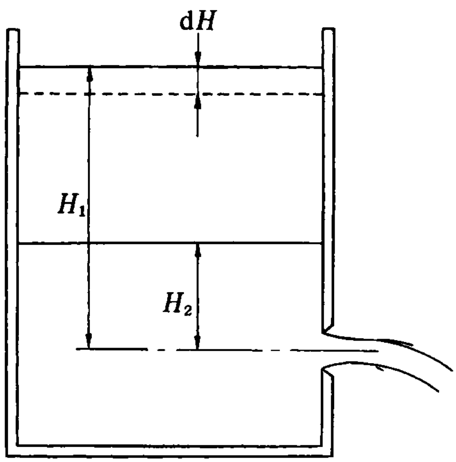


Fig. 5.12 Flow through a small hole (2)

Assume that dQ of water flows out in time dt with the water level falling by $-dH$ (Fig. 5.12). Then

$$dQ = Ca\sqrt{2gH}\,dt = -dH A$$

$$dt = \frac{-A\,dH}{Ca\sqrt{2gH}}$$

$$\int_{t_1}^{t_2} dt = -\frac{A}{Ca\sqrt{2g}}\int_{H_1}^{H_2}\frac{dH}{\sqrt{H}}$$

The time needed for the water level to descend from H_1 to H_2 is

$$t_2 - t_1 = \frac{2A}{Ca\sqrt{2g}}(\sqrt{H_1} - \sqrt{H_2}) \tag{5.26}$$

Flow through a small hole 3: *the section of water tank where the descending velocity of the water level is constant*

Assume that the bottom has a small hole of area a, through which water flows (Fig. 5.13), then

$$dQ = Ca\sqrt{2gH}\,dt = -dH\,A = -dH\,\pi r^2$$

Whenever the descending velocity of the water level ($-dH/dt = v$) is constant, the above equation becomes

$$v = -\frac{dH}{dt} = \frac{Ca\sqrt{2gH}}{\pi r^2} \tag{5.27}$$

$$H = \left(\frac{\pi v}{Ca\sqrt{2g}}\right)^2 r^4 \tag{5.28}$$

$$H \propto r^4 \tag{5.29}$$

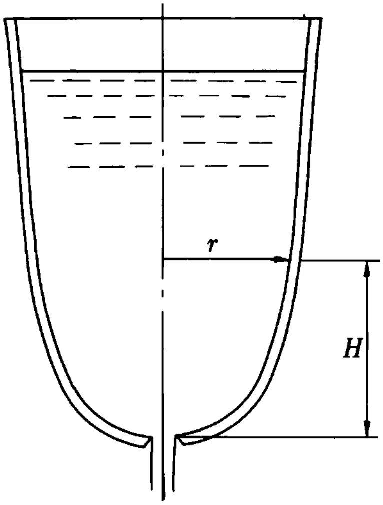


Fig. 5.13 Flow through a small hole (3)

In other words, whenever the section shape has a curve of r^4 against the vertical line, the descending velocity of the water level is constant.

Figure 5.14 shows a water clock made in Egypt about 3400 years ago, which indicates the time by the position of the water level.

Fig. 5.14 Egyptian water clock 3400 years old (London Science Museum)

Weir

As shown in Fig. 5.15, in the case where a water channel is stemmed by a board or a wall, over which the water flows, such a board or wall is called a weir. A weir is used to adjust the flow rate.

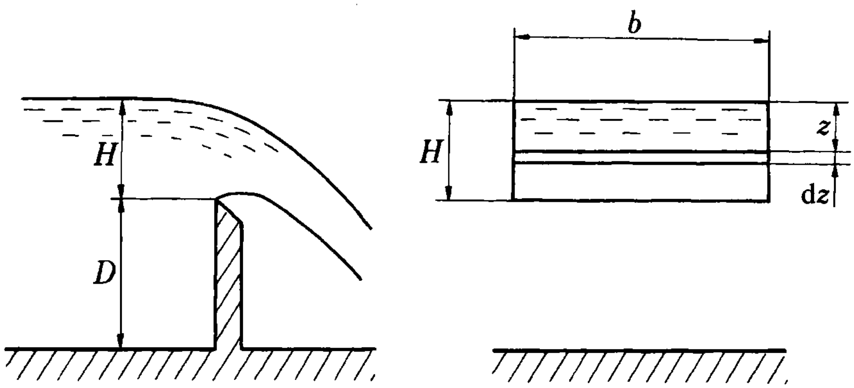


Fig. 5.15 Weir

In the figure, assume a minute depth dz at a given depth z from the water level. Let b be the width of the water channel and assume a minute area $b\,\mathrm{d}z$ as an orifice. From Bernoulli's equation

$$v = \sqrt{2gz}$$

The flow rate dQ passing here is as follows assuming the coefficient of discharge is C:

$$\mathrm{d}Q = Cb\,\mathrm{d}z\sqrt{2gz}$$

Integrating the above equation,

$$Q = Cb\sqrt{2g}\int_0^H \sqrt{z}\,\mathrm{d}z = \tfrac{2}{3}Cb\sqrt{2g}H^{3/2} \tag{5.30}$$

By measuring H, the discharge Q can be computed from eqn (5.30).

5.3 Conservation of momentum

5.3.1 Equation of momentum

A flying baseball can simply be caught with a glove. A moving automobile, however, is difficult to stop in a short time (Fig. 5.16). Therefore, the velocity is not sufficient to study the effects of bodily motion, but the product, Mv, of the mass M and the velocity v can be used as an indicator of the consequences of motion. This is called the linear momentum. By Newton's second law of motion, the change per unit time in the momentum of a body is equal to the force acting on the body.

Now, assume that a body of mass M (kg) will be at velocity v (m/s) in t seconds. The acting force F (N) is given by the following equation:

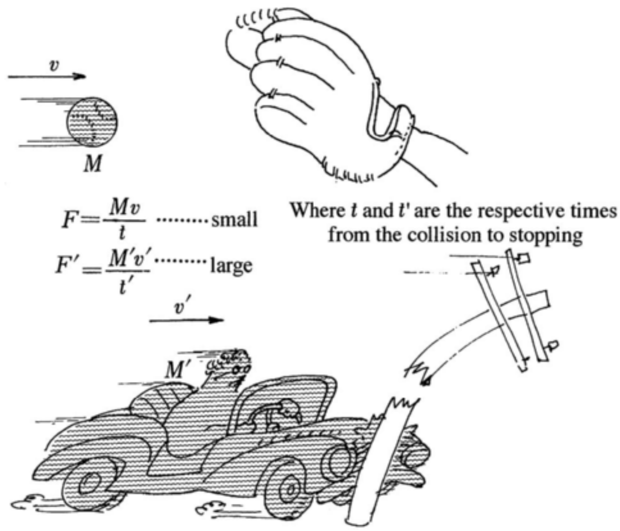


Fig. 5.16 Car does not stop immediately

$$F = \frac{Mv_2 - Mv_1}{t} \tag{5.31}$$

In other words, the acting force is conserved as an increase in unit time in momentum. This is the law of conservation of momentum.

Whenever the reaction force of a jet or the force acting on a solid wall in contact with the flow is to be obtained, by using the change in momentum, such a force can be obtained comparatively simply without examining the complex internal phenomena.

In an actual computation, keeping in mind an assumed control volume in the flow, the relation between the change in momentum and the force within that volume is obtained by using the equation of momentum. In the case where fluid flows in a curved pipe as shown in Fig. 5.17, let ABCD be the control volume, A_1, A_2 the areas, v_1, v_2 the velocities, and p_1, p_2 the pressures of sections AB and CD respectively. Furthermore, let F be the force of fluid acting on the pipe; the force of the pipe acting on the fluid is $-F$. This force and the pressures acting on sections AB and CD act on the fluid, increasing the fluid momentum by such a combined force.[1] If F_x and F_y are the component forces in the x and y directions of F respectively, then from the equation of momentum,

$$\left.\begin{aligned} -F_x + A_1p_1\cos\alpha_1 - A_2p_2\cos\alpha_2 &= m(v_2\cos\alpha_2 - v_1\cos\alpha_1) \\ -F_y + A_1p_1\sin\alpha_1 - A_2p_2\sin\alpha_2 &= m(v_2\sin\alpha_2 - v_1\sin\alpha_1) \end{aligned}\right\} \tag{5.32}$$

[1] Increase in momentum = momentum going out − momentum coming in.

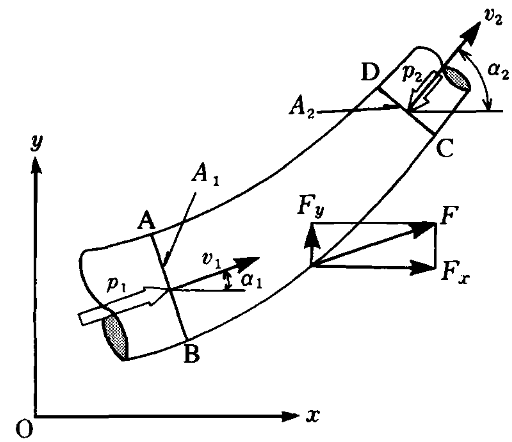


Fig. 5.17 Flow in a curved pipe

In this equation, m is the mass flow rate. If Q is the volumetric flow rate, then the following relation exists:

$$m = \rho Q = \rho A_1 v_1 = \rho A_2 v_2 = \rho Q$$

From eqn (5.32), F_x and F_y are given by

$$\left.\begin{aligned} F_x &= m(v_1 \cos\alpha_1 - v_2 \cos\alpha_2) + A_1 p_1 \cos\alpha_1 - A_2 p_2 \cos\alpha_2 \\ F_y &= m(v_1 \sin\alpha_1 - v_2 \sin\alpha_2) + A_1 p_1 \sin\alpha_1 - A_2 p_2 \sin\alpha_2 \end{aligned}\right\} \quad (5.33)$$

Equation (5.32) is in the form where the change in momentum is equal to the force, but since m refers to mass per unit time, note that the equation shows that the time-sequenced change in momentum is equal to the force.

The combined force acting on the curved pipe can be obtained by the following equation:

$$F = \sqrt{F_x^2 + F_y^2} \quad (5.34)$$

5.3.2 Application of equation of momentum

The equation of momentum is very effective when a fluid force acting on a body is studied.

Force of a jet

Let us study the case where, as shown in Fig. 5.18, a two-dimensional jet flow strikes an inclined flat plate at rest and breaks into upward and downward jets.

Assume that the internal pressure of the jet flow is equal to the external one and that no loss arises from the flow striking the flat plate. Since no loss occurs, it is assumed that the fluid flows out at the velocity v along the flat board after striking it. The control volume is conceived as shown in Fig. 5.18.

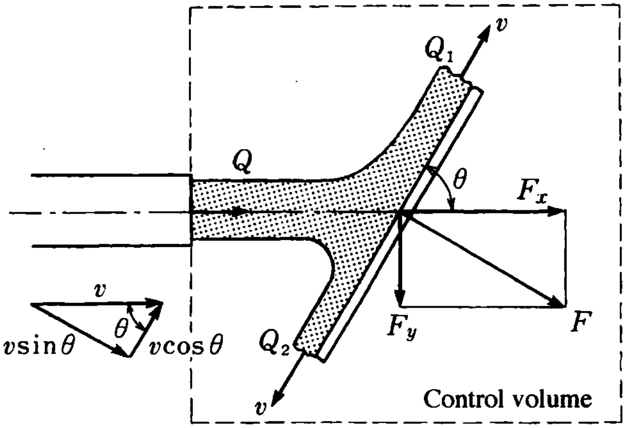


Fig. 5.18 Force of jet acting on a flat plate at rest

Examining the direction at right angles to the flat plate, since the velocity of the jet turns out to be zero after it has struck the flat board at $v \sin\theta$,

$$F = \rho Q v \sin\theta \tag{5.35}$$

Force F_x acting in the direction of the jet is

$$F_x = F \sin\theta = \rho Q v \sin^2\theta \tag{5.36}$$

Force F_y acting in the direction at right angles to the jet is

$$F_y = F \cos\theta = \rho Q v \sin\theta \cos\theta \tag{5.37}$$

Then the flow rate along the flat plate separates into Q_1 and Q_2. Let us obtain the change in the ratio of Q_1 to Q_2 according to the inclined angle θ. In this case, since no force acts along the flat board if the flow loss is disregarded, applying the equation of momentum to the direction along the flat board,

$$\rho Q v \cos\theta = \rho Q_1 v - \rho Q_2 v \quad Q \cos\theta = Q_1 - Q_2$$

Q_1 and Q_2 are obtained using the continuity equation $Q = Q_1 + Q_2$, and

$$Q_1 = Q(1 + \cos\theta)/2 \tag{5.38}$$

$$Q_2 = Q(1 - \cos\theta)/2 \tag{5.39}$$

In the case where the flat board in Fig. 5.18 moves in the same direction as the jet flow at velocity u, since the relative velocity of the jet flow compared with the flat board is $v - u$, the flow rate Q' reaching the flat board is given by

$$Q' = Q\frac{v - u}{v}$$

Since the change in velocity in the direction at right angles to the flat board is $(v - u)\sin\theta$, force F acting on the flat board is therefore

$$F = \rho Q'(v-u)\sin\theta = \rho Q\frac{(v-u)^2}{v}\sin\theta \tag{5.40}$$

Loss in a suddenly expanding pipe

For a suddenly expanding pipe as shown in Fig. 5.19, assume that the pipe is horizontal, disregard the frictional loss of the pipe, let h_s be the expansion loss, and set up an equation of energy between sections 1 and 2 as

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + h_s$$

or

$$h_s = \frac{p_1 - p_2}{\rho g} + \frac{v_1^2 - v_2^2}{2g} \tag{5.41}$$

Next, the streamlines in the smaller pipe are parallel at its very end, so the pressure there is p_1. And it can be considered that the pressure at the cross section is constant, so the pressure on the annular face at the pipe joint is also p_1. Apply the equation of momentum setting the control volume as shown in Fig. 5.19. Thus

$$\rho Q(v_2 - v_1) = (p_1 - p_2)A_2 \tag{5.42}$$

Since $Q = A_1 v_1 = A_2 v_2$, from the above equation,

$$\frac{p_1 - p_2}{\rho g} = \frac{Q}{A_2}\frac{v_2 - v_1}{g} = \frac{v_2}{g}(v_2 - v_1) \tag{5.43}$$

Substituting eqn (5.43) into (5.41),

$$h_s = \frac{(v_1 - v_2)^2}{2g} = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{v_1^2}{2g} \tag{5.44}$$

is obtained. This h_s is called the Borda–Carnot head loss or simply the expansion loss.

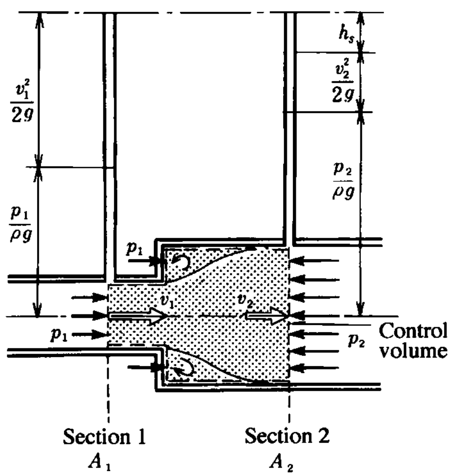


Fig. 5.19 Abruptly enlarging pipe

Jet pump

A jet pump is constructed as shown in Fig. 5.20. By making a water jet spout out into a larger water pipe, mixing with the surrounding water occurs so that it is carried out with that jet flow.

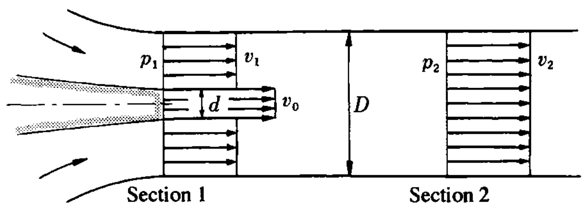


Fig. 5.20 Jet pump

If v_0 is the velocity of the jet discharging at section 1 and v_1 the velocity of the surrounding water, and assuming that mixing finishes at section 2 and the flow is then at uniform velocity v_2, then we have the following:

outflow momentum:	$\frac{\pi D^2}{4}\rho v_2^2$
inflow in momentum:	$\frac{\pi}{4}(D^2 - d^2)\rho v_1^2 + \frac{\pi}{4}d^2\rho v_0^2$
increase in momentum:	$\frac{\pi}{4}\rho\left[D^2 v_2^2 - (D^2 - d^2)v_1^2 - d^2 v_0^2\right]$
force acting on the fluid:	$\frac{\pi}{4}D^2(p_1 - p_2)$

By the law of momentum,

$$\rho\left[D^2 v_2^2 - (D^2 - d^2)v_1^2 - d^2 v_0^2\right] = D^2(p_1 - p_2)$$

Rearranging using the continuity equation,

$$p_2 - p_1 = \rho\frac{d^2}{D^2}\frac{D^2 - d^2}{D^2}(v_0 - v_1)^2 \tag{5.45}$$

This equation shows that $p_2 - p_1$ is always positive. In other words, a jet pump can force out water against the differential pressure.

Efficiency of a propeller

In the case shown in Fig. 5.21, a propeller of diameter D moving from right to left at velocity U can be considered as the case where a flow from left to right at velocity U strikes a propeller at rest. It can also be assumed that the fluid downstream has been accelerated to velocity $U + u$. Furthermore, the pressures upstream and downstream of the propeller are equally constant p.

From the changes in momentum and kinetic energy across the revolving face of the propeller, the thrust T is given by

$$T = \frac{\pi}{4}D^2\rho u\left(U + \frac{u}{2}\right) \tag{5.46}$$

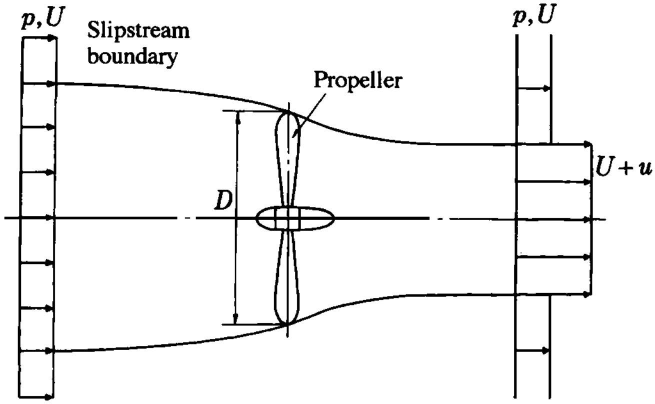


Fig. 5.21 Flows upstream and downstream of a propeller

and the efficiency η by

$$\eta = \frac{2}{2 + u/U} \tag{5.47}$$

Since the losses due to the fluid viscosity and the revolution of the wake are disregarded in this computation, this theory gives the attainable upper limit.

5.4 Conservation of angular momentum

5.4.1 Equation of angular momentum

The angular momentum in the case where a body of mass M is rotating at radius r and rotational velocity v is given by

$$\begin{aligned} \text{Angular momentum} &= \text{moment of inertia} \times \text{angular velocity} \\ &= Mr^2 \times \frac{v}{r} = Mrv \end{aligned} \tag{5.48}$$

The torque (rotational couple) on this body is given by

$$\begin{aligned} \text{Torque} &= \text{change of angular momentum} \\ &= \text{moment of inertia} \times \text{angular acceleration} \end{aligned} \tag{5.49}$$

This is equivalent to Newton's second law of motion, and expresses the law of conservation of angular momentum.

Figure 5.22 shows a diagram of an ice skater. Whenever the skater revolves with the same angular momentum, if she spreads out her arms and stretches out one of her legs to enlarge the moment of inertia, she will slow down. This graphically expresses the relation of eqn (5.49).

If the relation of eqn (5.49) is applied to fluid flow, the torque acting on

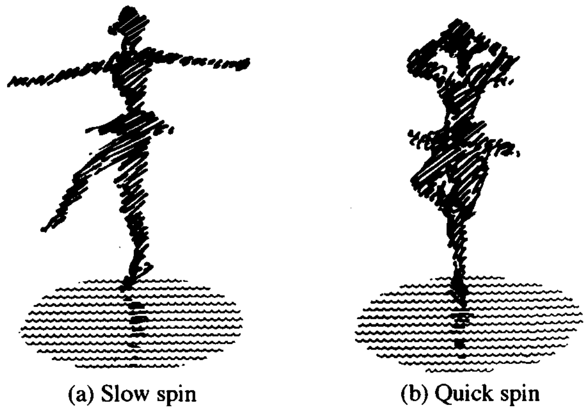


Fig. 5.22 Ice skater

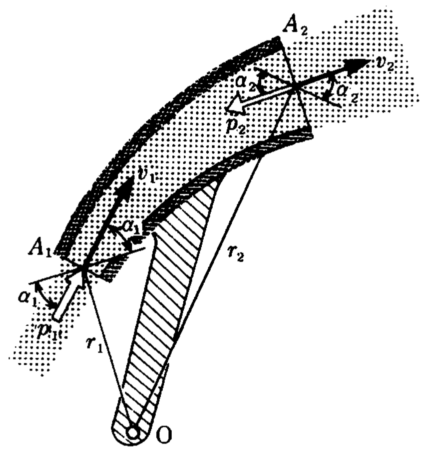


Fig. 5.23 Flow in curved tube supported so as to turn around shaft O

the shaft of a water wheel or a pump when the fluid runs over its rotating impeller can be obtained.

In the case where fluid is running in a curved tube as shown in Fig. 5.23, let T be the moment (torque),[2] which tries to turn the pipe around shaft O, generated by the force which the fluid between section A_1 and section A_2 exerts on the pipe wall. Then from the equation of angular momentum

$$T + A_2 p_2 r_2 \cos\alpha_2 - A_1 p_1 r_1 \cos\alpha_1 = m(r_2 v_2 \cos\alpha_2 - r_1 v_1 \cos\alpha_1) \tag{5.50}$$

[2] The directions of rotation and torque are usually positive whenever they are counterclockwise.

5.4.2 Power of a water wheel or pump

Fluid flows at mass flow rate m along the blade in Fig. 5.24 due to rotation of the pump impeller. At radii r_1, r_2, the peripheral velocities are u_1, u_2 and v_1, v_2 are the absolute velocities at angles α_1, α_2 to them. The relative velocities to the impeller are w_1 and w_2. As seen from Fig. 5.24, since the direction of the pressures passes through the centre of the impeller, the second and third terms on the left eqn (5.50) turn out to be zero. The torque is as follows:

$$T = m(r_2 v_2 \cos\alpha_2 - r_1 v_1 \cos\alpha_1) \tag{5.51}$$

In this way, the torque acting on the impeller shaft can be obtained just from the states of the velocities at the inlet and outlet of the impeller.

If ω is the angular velocity of the impeller, the power L given to the shaft is

$$L = T\omega \tag{5.52}$$

The torque and power for a water wheel can be obtained similarly.

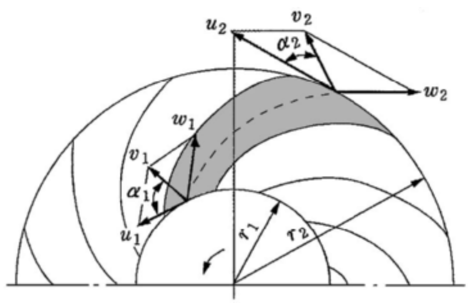


Fig. 5.24 Flow along blade of centrifugal pump

5.5 Problems

1. Derive Bernoulli's equation for steady flow by integrating Euler's equation of motion.

2. Find the flow velocities v_1, v_2 and v_3 in the conduit shown in Fig. 5.25. The flow rate Q is 800 L/min and the diameters d_1, d_2 and d_3 at sections 1, 2 and 3 are 50, 60 and 100 mm respectively.

3. Water is flowing in the conduit shown in Fig. 5.25. If the pressure p_1 at section 1 is 24.5 kPa, what are the pressures p_2 and p_3 at sections 2 and 3 respectively?

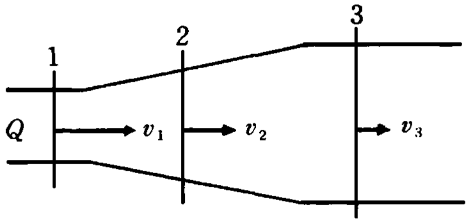


Fig. 5.25

4. In Fig. 5.26, air of flow rate Q flows into the centre through a pipe of radius r, and radially between two discs, and then flows out into the atmosphere. Obtain the pressure distribution between the discs. Also calculate the pressure force acting on the lower annular ring plate whose inner diameter is r_1 and outer diameter is r_2. Neglect frictional losses.

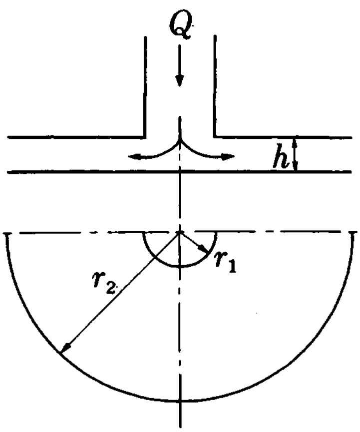


Fig. 5.26

5. In Fig. 5.26, if water flows at rate $Q = 0.013\,\text{m}^3/\text{s}$ radially between two discs of radius $r_2 = 30\,\text{cm}$ each from a pipe of radius $r_1 = 7\,\text{cm}$, obtain the pressure and the flow velocity at $r = 12\,\text{cm}$. Assume that $h = 0.3\,\text{cm}$ and neglect the frictional loss.

6. As shown in Fig. 5.27, a tank has a hole and $a \ll A$. Find the time necessary for the tank to empty.

7. As shown in Fig. 5.28, water flows out of a vessel through a small hole in the bottom. What is a suitable section shape to keep the velocity of descent of the water surface constant? Assume the volume of water in the vessel is 2 l, $R/d = 100$ (where R is the radius of the initial water surface in the vessel, d the small hole on the bottom), and the flow discharge coefficient of the small hole is $C = 0.6$. What should R and d be in order to manufacture a water clock for measuring 1 hour?

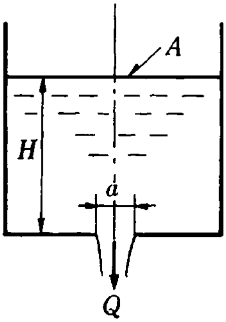


Fig. 5.27

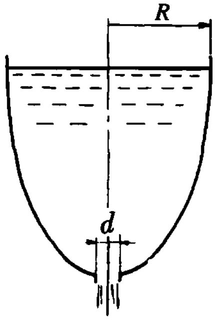


Fig. 5.28

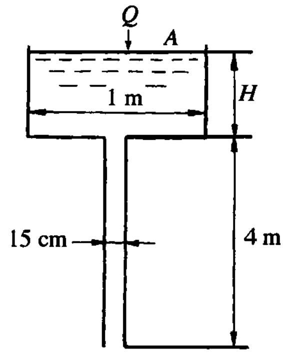


Fig. 5.29

8. In the case shown in Fig. 5.29, water at a flow rate of $Q = 0.2\,\mathrm{m^3/s}$ is supplied to the cylindrical water tank of diameter 1 m discharging through a round pipe of length 4 m and diameter 15 cm. How deep will the water in the tank be?

9. As shown in Fig. 5.30, a jet of water of flow rate Q and diameter d strikes the stationary plate at angle θ. Calculate the force on this stationary plate and its direction. Furthermore, if $\theta = 60°$, $d = 25\,\mathrm{mm}$ and $Q = 0.12\,\mathrm{m^3/s}$, obtain Q_1, Q_2 and F.

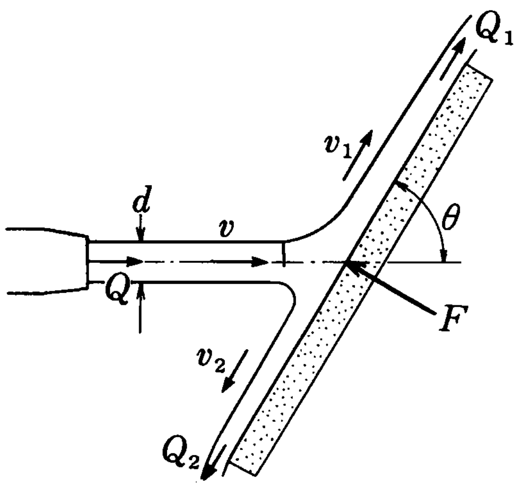


Fig. 5.30

10. As shown in Fig. 5.31, if water flows out of the tank of head 50 cm through the throttle, obtain the pressure at the throat.

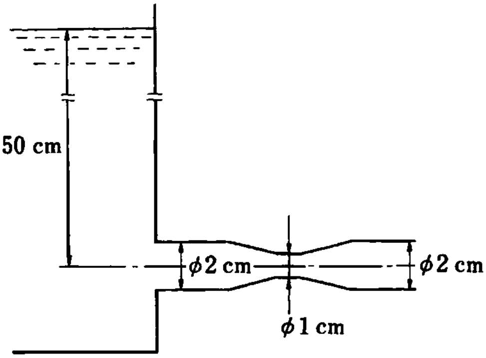


Fig. 5.31

11. Figure 5.32 shows a garden sprinkler. If the sprinkler nozzle diameter is 5 mm and the sprinkler velocity is 5 m/s, what is the rate of rotation? What torque is required to hold the sprinkler stationary? Assume there is no friction.

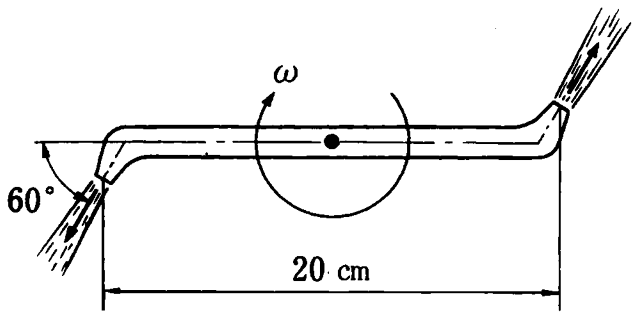


Fig. 5.32

12. A jet-propelled boat as shown in Fig. 5.33 is moving at a velocity of 10 m/s. The river is flowing against the boat at 5 m/s. Assuming the jet flow rate is 0.15 m^3/s and its discharge velocity is 20 m/s, what is the propelling power of this boat? (Jet boats like this are actually in use.)

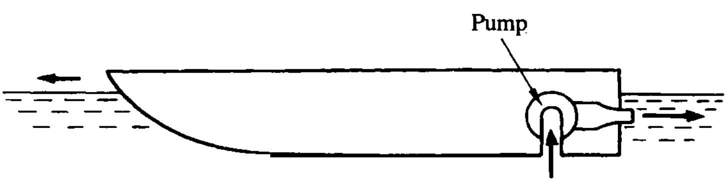


Fig. 5.33